

ON THE DYNAMICS OF CODIMENSION ONE HOLOMORPHIC FOLIATIONS WITH AMPLE NORMAL BUNDLE

MARCO BRUNELLA

ABSTRACT. We investigate the accumulation to singular points of leaves of codimension one foliations whose normal bundle is ample, with emphasis on the nonexistence of Levi-flat hypersurfaces.

1. INTRODUCTION

Let X be a compact complex manifold and let \mathcal{F} be a (singular) codimension one holomorphic foliation on X . From the dynamical point of view, one of the most basic problems we may address is the following [CLS]: does it hold that every leaf of \mathcal{F} accumulates to the singular set $Sing(\mathcal{F})$? Of course, for a positive answer we need some hypothesis on X and/or \mathcal{F} . In [Lin], Lins Neto showed that the answer to the above question is positive when X is the complex projective space $\mathbb{C}P^n$ and $n \geq 3$ (the case $n = 2$ is still open, and much more difficult). Our aim is to prove a similar result on any manifold X , under some assumption on \mathcal{F} .

Recall that we may associate to \mathcal{F} its *normal bundle* $N_{\mathcal{F}}$: the foliation is locally defined by integrable holomorphic 1-forms $\omega_j \in \Omega^1(U_j)$, with zero set of codimension at least two, and so $N_{\mathcal{F}}$ is the holomorphic line bundle defined by the cocycle $\{g_{jk} \in \mathcal{O}^*(U_j \cap U_k)\}$ given by $\omega_j = g_{jk}\omega_k$. Our feeling is that $N_{\mathcal{F}}$ should reflect some dynamical properties of \mathcal{F} . A first instance of this philosophy is the result of [BLM] concerning the existence of loops with hyperbolic holonomy: by an easy “regularisation” of their proof, one can see that such a result holds under the sole assumption that $N_{\mathcal{F}}$ is *ample*. Other results involving some form of positivity of $N_{\mathcal{F}}$ can be found in [Der] and [Bru].

Let us put forward a conjecture.

Conjecture 1.1. *Let X be a compact connected complex manifold of dimension $n \geq 3$, and let \mathcal{F} be a codimension one holomorphic foliation on X whose normal bundle $N_{\mathcal{F}}$ is ample. Then every leaf of \mathcal{F} accumulates to $Sing(\mathcal{F})$.*

Note that by Baum-Bott formula [Suw] the ampleness of $N_{\mathcal{F}}$ implies, at least, that $Sing(\mathcal{F})$ is not empty; we shall return later on this point.

If $X = \mathbb{C}P^n$, or more generally if X admits an hermitian metric of positive curvature, then the ampleness hypothesis is automatically satisfied, because $N_{\mathcal{F}}$ is, outside the singular set, a quotient of TX and therefore it is more positive than TX . Hence the above Conjecture would extend the result of [Lin]. However, there are many cases in which we can guarantee that $N_{\mathcal{F}}$ is ample even if TX is far from positive, for example when X is a complete intersection (of dimension at least 3) in $\mathbb{C}P^N$ (indeed, $Pic(X) = \mathbb{Z}$ by Lefschetz and thus either $N_{\mathcal{F}}$ is ample or $N_{\mathcal{F}}^*$ is effective, but the latter is excluded by $\Omega^1(X) = \{0\}$).

Unfortunately, we are unable to prove the above Conjecture in full generality. Suppose that Conjecture 1.1 does not hold: then there exists a nonempty compact subset $\mathcal{M} \subset X$ which is invariant by \mathcal{F} and disjoint from $\text{Sing}(\mathcal{F})$. We shall prove that such a \mathcal{M} cannot be a sufficiently smooth real hypersurface.

More precisely, and more generally:

Theorem 1.1. *Let X be a compact connected Kähler manifold of dimension $n \geq 3$. Let $M \subset X$ be a closed real hypersurface of class $C^{2,\alpha}$, $\alpha > 0$. Suppose that on some neighbourhood U of M there exists a codimension one holomorphic foliation \mathcal{F} which leaves invariant M . Then the normal bundle $N_{\mathcal{F}}$ does not admit, on U , any hermitian metric with positive curvature.*

It is perhaps worth noting that, in the situation of this Theorem, the foliation \mathcal{F} is necessarily nonsingular around M .

We stated the result in a slightly cumbersome way because we hope that it could be generalised to a larger context, by “shrinking U to M ”, in the same way as [Siu] is a generalisation of [Lin]. That is, suppose that $M \subset X$ is a sufficiently smooth Levi-flat hypersurface (i.e., a real hypersurface smoothly foliated by complex hypersurfaces), but not necessarily invariant by a holomorphic foliation on a neighbourhood of it. It is still possible to define a normal bundle $N_{\mathcal{F}_M}$ of the Levi foliation \mathcal{F}_M on M . It is a smooth \mathbb{C} -bundle on M , holomorphic along the leaves; generally speaking, $N_{\mathcal{F}_M}$ cannot be extended to a holomorphic line bundle on a neighbourhood of M , in the same way as \mathcal{F}_M cannot be extended to a holomorphic foliation [BdB]. However, it makes sense to say that $N_{\mathcal{F}_M}$ admits a hermitian metric of positive curvature along the leaves of \mathcal{F}_M . We think that it is *never* the case.

When $X = \mathbb{C}P^n$ such a generalisation has been done by Siu in [Siu]; in that case $N_{\mathcal{F}_M}$ *does* admit a metric with leafwise positive curvature, by quotient of the Fubini-Study metric, and so the conclusion is that M does not exist.

Remark that if M is real analytic then \mathcal{F}_M and $N_{\mathcal{F}_M}$ can always be holomorphically extended to some neighbourhood U of M , and a metric on $N_{\mathcal{F}_M}$ with leafwise positive curvature can always be extended to a metric on the extended bundle with positive curvature: the positivity in the direction transverse to the leaves can be gained by multiplying any extended metric by the factor $\exp[-C \text{dist}(\cdot, M)^2]$, $C \gg 0$. Thus we return to the setting of Theorem 1.1.

If M is of class C^∞ , \mathcal{F}_M can be extended at least as a “formal” object, i.e. as a C^∞ -section of the bundle of hyperplanes of X whose $\bar{\partial}$ vanishes along M at infinite order. Optimistically, one could try to construct a holomorphic extension using a sort of formal-convergent principle *à la* Hironaka-Matsumura, and the leafwise positivity of $N_{\mathcal{F}_M}$. This is related to the vanishing theorem proved in [Siu] and [Bri], and it remains to see if that vanishing theorem can be applied in our context.

Similar problems can be posed also in the context of Conjecture 1.1. That is, one may ask about the existence of codimension one laminations (not necessarily invariant by a holomorphic foliation) whose normal bundle is leafwise ample. Even in the case of $X = \mathbb{C}P^n$, this seems a still open problem.

In another direction, we hope that Theorem 1.1 could be useful to classify Levi-flat hypersurfaces in Fano manifolds, at least under the assumption of invariance by a global holomorphic foliation. Indeed, if X is a Fano manifold (i.e., its anticanonical bundle K_X^* is ample), then $N_{\mathcal{F}} = K_X^* \otimes K_{\mathcal{F}}$ is more positive than the canonical bundle $K_{\mathcal{F}}$ of the foliation. Thus, if $M \subset X$ is a real hypersurface of class $C^{2,\alpha}$

invariant by \mathcal{F} , then $K_{\mathcal{F}}$ cannot be nef, by Theorem 1.1 (recall that nef + ample is ample). According to Miyaoka and Shepherd-Barron [ShB], this gives remarkable informations on \mathcal{F} , in particular concerning the existence of rational curves inside the leaves. Eventually, all of this could prove that M is smoothly fibered by compact rationally connected submanifolds of dimension $n - 2$, contained in the leaves. In a different context, the one of complex tori, a similar structure has been found by Ohsawa [Ohs].

Our proof of Theorem 1.1 follows the same path as [Lin]. We proceed by contradiction, by assuming that $N_{\mathcal{F}}$ has a metric of positive curvature. The main step consists in proving that the complement $X \setminus M$ is then *strongly pseudoconvex*, that is a point modification of a Stein space. This step is by free in [Lin], in an even stronger form, thanks to the classical solution of the Levi problem in projective spaces [Fuj] [Tak]. Of course, Levi problem has a negative answer on most compact Kähler manifolds, and in our case we need a delicate glueing procedure of plurisubharmonic functions, which involves the $C^{2,\alpha}$ -regularity of M . Then we conclude the proof in two slightly different ways, one close to [Siu] and the other close to [Lin]. It is only in this last part that the dimensional hypothesis $n \geq 3$ is used.

In the last Section we discuss some possible ways to fill the gap between Theorem 1.1 and Conjecture 1.1.

2. CONVEXITY OF THE COMPLEMENT

Recall that a complex manifold V is *strongly pseudoconvex* (or *1-convex*) if there exists a C^2 function $\psi : V \rightarrow \mathbb{R}$ which is:

- (i) exhaustive, i.e. $\{\psi \leq c\}$ is compact for every $c \in \mathbb{R}$;
- (ii) strictly plurisubharmonic, i.e. $i\partial\bar{\partial}\psi > 0$, outside a compact subset.

Classical results by Grauert and Remmert [Pet, §2] say that a strongly pseudoconvex manifold V is a point modification of a Stein space V_0 : there exists a proper holomorphic map $\pi : V \rightarrow V_0$ and a finite subset $P \subset V_0$ such that π is an isomorphism between $V \setminus \pi^{-1}(P)$ and $V_0 \setminus P$. The *exceptional subset* $\pi^{-1}(P)$ is the maximal compact analytic subset of V of positive dimension.

Before starting the proof of Theorem 1.1, let us recall the well-known and easy proof of the following model case: if X is a compact connected complex manifold and $Y \subset X$ is a (smooth) complex hypersurface whose normal bundle N_Y is ample (on Y), then $X \setminus Y$ is strongly pseudoconvex. Indeed, by the adjunction formula N_Y may be identified with $\mathcal{O}_X(Y)|_Y$, and so we may construct on $\mathcal{O}_X(Y)$ a hermitian metric whose curvature is positive on some neighbourhood of Y . The line bundle $\mathcal{O}_X(Y)$ has a global holomorphic section s on X , which vanishes exactly on Y . Then the function $\psi = -\log \|s\|$, where $\|\cdot\|$ is the above metric, is an exhaustion of $X \setminus Y$ and is strictly plurisubharmonic close to Y . Here $\|s\|$ plays the role of distance from Y , and below we shall need to find a good substitute for it in our context. See also [Tak] and [Ohs] (without mentioning Oka...).

Consider X , M , \mathcal{F} as in Theorem 1.1, and suppose by contradiction that $N_{\mathcal{F}}$ has, on some neighbourhood U of M , a hermitian metric with positive curvature ω . In this Section we shall prove:

Proposition 2.1. *$X \setminus M$ is strongly pseudoconvex.*

We may assume that M is connected and, up to taking a double covering, orientable; hence, up to reducing U , $U \setminus M$ has two connected components U^+ and U^- .

We may choose U so that it is covered by a finite number of charts $U_j \simeq \mathbb{D} \times \mathbb{D}^{n-1}$ adapted to the foliation, $j = 1, \dots, \ell$, and M cuts each U_j along $M_j = \gamma_j \times \mathbb{D}^{n-1}$, where $\gamma_j \subset \mathbb{D}$ is a proper arc of class $C^{2,\alpha}$. Thus $U_j \setminus M_j$ has two connected components U_j^+ and U_j^- , contained respectively in U^+ and U^- . We have $U_j^+ = V_j^+ \times \mathbb{D}^{n-1}$ and $U_j^- = V_j^- \times \mathbb{D}^{n-1}$, where V_j^+ and V_j^- are the two connected components of $\mathbb{D} \setminus \gamma_j$.

For each $j = 1, \dots, \ell$, choose a biholomorphism

$$\varphi_j : V_j^+ \longrightarrow \mathbb{D}^+ = \{z \in \mathbb{D} \mid \Im m z > 0\}$$

sending the arc $\gamma_j \subset \partial V_j^+$ to the arc $(-1, 1) \subset \partial \mathbb{D}^+$. This is possible thanks to a classical result by Carathéodory [Pom, §2] concerning the boundary extension of conformal maps between Jordan domains. Moreover, thanks to an as much as classical result by Kellogg and Warschawski [Pom, §3], the map φ_j (and its inverse) is of class $C^{2,\alpha}$ up to the boundary. More precisely, φ_j extends to a $C^{2,\alpha}$ -diffeomorphism between $V_j^+ \cup \gamma_j$ and $\mathbb{D}^+ \cup (-1, 1)$. In the following we shall need only the C^2 (or even $C^{1,1}$) regularity, but generally speaking Kellogg-Warschawski's theorem does not hold in the limit case $\alpha = 0$, whence our assumption $M \in C^{2,\alpha}$ instead of $M \in C^2$.

Then we define

$$f_j = \varphi_j \circ \pi_j : U_j^+ \longrightarrow \mathbb{D}^+$$

where $\pi_j : U_j \rightarrow \mathbb{D}$ is the natural projection along the leaves. Hence f_j is holomorphic in U_j^+ and of class C^2 up to $M_j \subset \partial U_j^+$. Its differential df_j is a holomorphic 1-form on U_j^+ vanishing on \mathcal{F} , thus a holomorphic section of $N_{\mathcal{F}}^*$ over U_j^+ . As a section of $N_{\mathcal{F}}^*$, df_j is nowhere vanishing in U_j^+ . Moreover, it extends to M_j as a section of class C^1 , and also on M_j it is nowhere vanishing.

Finally we define

$$h_j = \log \left\{ \frac{\|df_j\|}{\Im m f_j} \right\} : U_j^+ \longrightarrow \mathbb{R}$$

where $\|df_j\|$ is computed using the dual norm on $N_{\mathcal{F}}^*$, induced by the norm with positive curvature on $N_{\mathcal{F}}$.

Let us list some properties of these functions h_j .

(1) First of all, h_j is well defined because df_j does not vanish on U_j^+ . Moreover, $\Im m f_j$ tends to zero when approaching M_j , whereas $\|df_j\|$ has a finite nonzero limit, hence

$$h_j(p) \rightarrow +\infty \quad \text{as } p \rightarrow M_j.$$

(2) We may write $h_j = \log \|df_j\| - \log(\Im m f_j)$ and observe that the second term is plurisubharmonic whereas the $i\partial\bar{\partial}$ of the first term equals the curvature ω of $N_{\mathcal{F}}$, therefore

$$i\partial\bar{\partial}h_j \geq \omega.$$

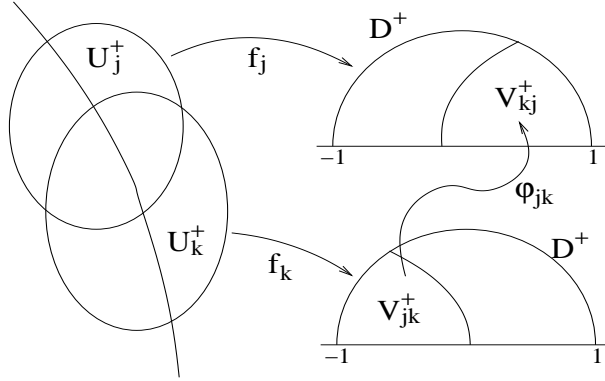
Now we want to glue the functions h_j , and so we need to estimate their differences $h_j - h_k$ on $U_j^+ \cap U_k^+$.

Set $V_{jk}^+ = f_k(U_j^+ \cap U_k^+) \subset \mathbb{D}^+$. Then, for every $j, k = 1, \dots, \ell$, we have a biholomorphism

$$\varphi_{jk} : V_{jk}^+ \longrightarrow V_{kj}^+$$

such that

$$f_j = \varphi_{jk} \circ f_k \quad \text{on } U_j^+ \cap U_k^+.$$



Hence $df_j = (\varphi'_{jk} \circ f_k) \cdot df_k$ and $\Im m f_j = (\Im m \varphi_{jk}) \circ f_k = (\frac{\Im m \varphi_{jk}}{\Im m} \circ f_k) \cdot \Im m f_k$, and so

$$h_j - h_k = \log \left\{ |\varphi'_{jk}| \cdot \frac{\Im m}{\Im m \varphi_{jk}} \right\} \circ f_k.$$

Now, by Schwarz reflection the biholomorphism φ_{jk} extends beyond the parts of the boundaries of V_{jk}^+ and V_{kj}^+ along $(-1, 1)$. Moreover, this extension sends the real axis into itself, and so φ'_{jk} is real on $(-1, 1)$ and equal to $\frac{\Im m \varphi_{jk}}{\Im m}$ there. It follows that the function $\log \left\{ |\varphi'_{jk}| \cdot \frac{\Im m}{\Im m \varphi_{jk}} \right\}$ appearing in the above expression of $h_j - h_k$ is real analytic up to $\partial V_{jk}^+ \cap (-1, 1)$ and equal to 0 there. As a consequence of this:

(3)

$$h_j(p) - h_k(p) \rightarrow 0 \quad \text{as } p \rightarrow M_j \cap M_k.$$

(4)

$$dh_j(p) - dh_k(p) \text{ stays bounded as } p \rightarrow M_j \cap M_k.$$

Of course, all these estimates hold in a uniform way. Note that in (4) we use the fact that f_k is of class C^1 up to M_k .

We are now ready to construct our exhaustion of $X \setminus M$.

Take a partition of unity $\{\psi_j\}_{j=1}^\ell$ adapted to $\{U_j\}_{j=1}^\ell$: each ψ_j is a nonnegative smooth function with compact support in U_j , and $\sum_{j=1}^\ell \psi_j \equiv 1$ around M . Define

$$h = \sum_{j=1}^\ell \psi_j h_j : U^+ \longrightarrow \mathbb{R}.$$

We obviously have $h(p) \rightarrow +\infty$ as $p \rightarrow M$ (uniformly), by property (1) above.

Lemma 2.1. *On a sufficiently small neighbourhood of M , the function h has no critical point, and the Kernel $\ker(\bar{\partial}h)_p \subset T_p X$ of $(\bar{\partial}h)_p$ uniformly converges to $T_q^\mathbb{C} M = T_q \mathcal{F}$ as $p \rightarrow q \in M$.*

Proof. Let us work, to fix notation, in the chart U_ℓ . The $\bar{\partial}$ -derivative of h can be written as

$$\bar{\partial}h = \sum_{j=1}^\ell \psi_j \bar{\partial}h_j + \sum_{j=1}^{\ell-1} (h_j - h_\ell) \bar{\partial}\psi_j.$$

The second term tends to 0 as p tends to M , by property (3). For the first term, we compute

$$\bar{\partial}h_j = \frac{\bar{\partial}\|df_j\|}{\|df_j\|} - \frac{\bar{\partial}(\Im m f_j)}{(\Im m f_j)}.$$

The 1-form $(\frac{\bar{\partial}\|df_j\|}{\|df_j\|})_p$ stays bounded as p tends to M , because f_j is of class C^2 up to M and df_j does not vanish on M . The 1-form $(\frac{\bar{\partial}(\Im m f_j)}{(\Im m f_j)})_p$ is, on the contrary, divergent as $p \rightarrow M$, and moreover its Kernel coincides with the one of df_j , i.e. $T_p\mathcal{F}$. It follows that $\ker(\bar{\partial}h)_p$ becomes closer and closer to $T_p\mathcal{F}$ as p approaches to M . \square

Now we compute the Levi form of h (still in the chart U_ℓ):

$$\begin{aligned} i\partial\bar{\partial}h &= \sum_{j=1}^{\ell} \psi_j i\partial\bar{\partial}h_j + \sum_{j=1}^{\ell-1} (h_j - h_\ell) i\partial\bar{\partial}\psi_j + \sum_{j=1}^{\ell-1} i\partial(h_j - h_\ell) \wedge \bar{\partial}\psi_j + \sum_{j=1}^{\ell-1} i\partial\psi_j \wedge \bar{\partial}(h_j - h_\ell) = \\ &= A + B + C + \bar{C}. \end{aligned}$$

By property (2) we have $A \geq \omega$, and by property (3) we have $B_p \rightarrow 0$ as $p \rightarrow M$. By property (4), the 2-form C_p is bounded as $p \rightarrow M$. Moreover, C_p vanishes on $T_p\mathcal{F}$, because $\partial(h_j - h_\ell)$ is proportional to df_ℓ (see the computation above). By this and by the previous Lemma, $C_p|_{\ker(\bar{\partial}h)_p}$ tends to 0 as $p \rightarrow M$.

Therefore, on a sufficiently small neighbourhood of M we certainly have

$$i\partial\bar{\partial}h|_{\ker(\bar{\partial}h)} > \frac{1}{2}\omega|_{\ker(\bar{\partial}h)}.$$

In other words, the (smooth) hypersurfaces $\{h = c\}$, $c \gg 0$, are strictly pseudoconvex. It is then easy to find a convex increasing r such that $r \circ h$ is strictly plurisubharmonic. After doing the analogous construction on the negative side U^- , we obtain our desired exhaustion of $X \setminus M$, strictly plurisubharmonic outside a compact subset.

Remark 2.1. The Kähler assumption has not been used up to now, nor the dimensional assumption $n \geq 3$.

3. END OF PROOF

Once we know that $X \setminus M$ is strongly pseudoconvex, and $\dim X \geq 3$, the proof can be concluded in several ways [Lin] [Siu].

First of all, we observe that $N_{\mathcal{F}}|_M$ is *topologically* trivial, because it has a nonvanishing section given by the “unit normal to M ” (as before, we may assume that M is orientable). Hence, the closed (1,1)-form ω is exact on M , as well as on a small tubular neighbourhood U of it; we may assume that ∂U is strictly pseudoconvex (from the exterior $X \setminus \bar{U}$), by Proposition 2.1.

Thus

$$\omega|_U = d\beta = \partial\beta^{0,1} + \bar{\partial}\beta^{1,0}$$

where the primitive $\beta = \beta^{0,1} + \beta^{1,0} \in A^1(U)$ can be chosen of real type ($\beta^{1,0} = \overline{\beta^{0,1}}$) and, from $d\beta = (d\beta)^{1,1}$,

$$\bar{\partial}\beta^{0,1} = 0.$$

According to a theorem of Grauert and Riemenschneider [Pet, Th. 5.8], and because $n = \dim X \geq 3$, the second cohomology group of the strongly pseudoconvex manifold $X \setminus M$ with \mathcal{O} -coefficients and with compact support $H_{\text{cpt}}^2(X \setminus M, \mathcal{O})$ is

equal to zero (indeed, by Serre's duality this group is isomorphic to the trivial $H^{n-2}(X \setminus M, K_X)$). This means that the $\bar{\partial}$ -closed $(0,1)$ -form $\beta^{0,1}$, defined on U , can be extended to the full X , as a $\bar{\partial}$ -closed $(0,1)$ -form $\tilde{\beta}^{0,1}$: firstly we extend in any (non $\bar{\partial}$ -closed) way, and then we correct the error using the vanishing of cohomology with compact support.

Because X is Kähler, so that $H^1(X, \mathcal{O}) \simeq H^0(X, \Omega^1)$ by complex conjugation, we may decompose

$$\tilde{\beta}^{0,1} = \bar{\eta} + \bar{\partial}\Phi$$

with $\eta \in \Omega^1(X)$ and $\Phi \in C^\infty(X)$. Hence $\partial\tilde{\beta}^{0,1} = \partial\bar{\partial}\Phi$ and therefore, setting $\Psi = i(\bar{\Phi} - \Phi)$:

$$\omega|_{U'} = i\partial\bar{\partial}\Psi.$$

Thus, we have found a strictly plurisubharmonic function on a neighbourhood of M . But this gives a contradiction with the maximum principle: a maximum point p for $\Psi|_M$ is also a maximum point for $\Psi|_L$, where L is the leaf through p , and this cannot exist.

This end of proof is very close to [Siu]. Really, all the difficulty of [Siu] is in the fact that there the form ω is defined only on M , and a priori it is not clear how to extend ω , as a closed $(1,1)$ -form, to a neighbourhood of M . Hence in Siu's paper the $(0,2)$ -form $\bar{\partial}\beta^{0,1}$ is not identically zero, as in our case, but it is only vanishing along M at some order (depending on the regularity of M). Thus, whereas we used basically only the resolubility of the $\bar{\partial}$ -equation with compact support to pass from $\beta^{0,1}$ to $\tilde{\beta}^{0,1}$, Siu needs a more delicate result (proved by himself) on the resolubility of the $\bar{\partial}$ -equation with growth conditions (see also [Bri]). In our case, as well as in [Lin], these difficulties disappear because, by assumption, the Levi foliation on M can be holomorphically extended to a neighbourhood of M , and this provides the required extension of ω .

Let us return a moment to the smooth case mentioned in the Introduction, in absence of a holomorphic foliation. As in [Siu] and [Bri], the nonexistence problem is reduced to construct a plurisubharmonic exhaustion of $X \setminus M$ with some additional good properties: this permits to prove a suitable vanishing theorem and then to repeat the arguments above (or, alternatively, to extend holomorphically the Levi foliation). The plurisubharmonic exhaustion that we constructed in the previous Section fits into this general scheme.

A slightly different end of proof is the following one, closer to the "topological" arguments of [Lin].

Consider the exceptional subset Y of the strongly pseudoconvex manifold $X \setminus M$. We may find an exhaustion $\psi : X \setminus M \rightarrow \mathbb{R}$ which is strictly plurisubharmonic outside Y . The classical Morse-type argument of Andreotti-Frankel-Thom allows to push any compact real surface in $X \setminus Y$ to a neighbourhood of M , using the gradient flow of ψ (and $n \geq 3$). In other words, $H^2(M, \mathbb{R})$ is isomorphic to $H^2(X \setminus Y, \mathbb{R})$.

Hence the closed $(1,1)$ -form ω (which, as before, can be extended to the full X , by pseudoconvexity) is exact not only on U but even on $X \setminus Y$. By the $\partial\bar{\partial}$ -lemma, we therefore obtain

$$\omega = \sum_{j=1}^m \lambda_j \delta_{Y_j} + i\partial\bar{\partial}T$$

where $\{Y_j\}_{j=1}^m$ are the irreducible components of Y of codimension one, λ_j are complex numbers, and T is a suitable current, smooth outside $\cup_{j=1}^m Y_j$. In particular, around M we have $\omega = i\partial\bar{\partial}T$, and we conclude as before by the maximum principle.

In fact, this second proof is equivalent to the first one: we have simply replaced the Hodge symmetry by the $\partial\bar{\partial}$ -lemma, but the former is also a consequence of the latter.

4. SOME MORE REMARKS

In trying to extend the previous proof of Theorem 1.1 to the more general context of Conjecture 1.1, one is faced with two main difficulties.

Suppose that Conjecture 1.1 does not hold, and so let $\mathcal{M} \subset X$ be a compact subset invariant by \mathcal{F} and disjoint from $Sing(\mathcal{F})$. We would like to prove that, thanks to the ampleness of $N_{\mathcal{F}}$, $X \setminus \mathcal{M}$ is strongly pseudoconvex.

We cover \mathcal{M} by charts U_j , where \mathcal{F} is defined by $f_j : U_j \rightarrow V_j \subset \mathbb{C}$. Then, on each $U_j \setminus \mathcal{M}_j$, with $\mathcal{M}_j = \mathcal{M} \cap U_j$, we take the function

$$h_j = \log \left\{ \frac{\|df_j\|}{dist_j(\cdot, \mathcal{M}_j)} \right\}$$

where $dist_j(\cdot, \mathcal{M}_j)$ is the “transverse” distance from \mathcal{M}_j , measured with f_j , that is

$$dist_j(p, \mathcal{M}_j) = \inf_{q \in \mathcal{M}_j} |f_j(p) - f_j(q)|.$$

The functions h_j that we used in Section 2 should be understood as special regularisations of these functions h_j .

It is easily checked that these $\{h_j\}$ satisfy properties similar to (1), (2) and (3) of Section 2. However, property (4) is a more delicate matter, due to the irregular behaviour of $dist_j(\cdot, \mathcal{M}_j)$. Let us see an example.

Example 4.1. Take, in the disc \mathbb{D} , the closed subset $K = \{\arg z = 0 \text{ or } \arg z = \frac{\pi}{2}\} \cup \{0\}$. Take two holomorphic embeddings $f_1, f_2 : \mathbb{D} \rightarrow \mathbb{C}$, and let $g_1, g_2 : \mathbb{D} \rightarrow \mathbb{R}$ be the corresponding distance functions from K . Each g_j is not C^1 along an arc $\gamma_j \subset \mathbb{D}$ starting at 0 with a tangent of argument $\frac{\pi}{4}$, the equidistant arc from the two branches of K . Typically, these arcs γ_1 and γ_2 bound (near the origin) a sector Ω adherent to 0, over which the logarithmic differentials of g_1 and g_2 are very far each other: one of them is close to $\frac{dx}{x}$, the other is close to $\frac{dy}{y}$. Thus $d \log g_1 - d \log g_2$ is unbounded, on any neighbourhood of 0.

If we replace K by a curve K' of class $C^{2,\alpha}$, the situation is not much better: g_j^2 are then of class $C^{2,\alpha}$ up to K' , but their quotient is probably no more than C^α along K' , and we don't see how to bound $d \log g_1 - d \log g_2$.

Related problems appear in trying to extend Lemma 2.1. However, our glueing procedure is rather rudimentary, and one could suspect that a more refined glueing procedure would work under the sole assumptions (1), (2), (3) of Section 2.

Suppose now that, in some way, we have proved that $X \setminus \mathcal{M}$ is strongly pseudoconvex. The second difficulty is that in Section 3 we used the fact that the 2-form ω , representing $N_{\mathcal{F}}$, is exact around M . We don't know if such a fact holds for a more general \mathcal{M} . But [Lin] suggests an alternative approach.

By Baum-Bott formula [Suw], the cohomology class $c_1^2(N_{\mathcal{F}})$ is localized in $Z = Sing(\mathcal{F})$. More precisely, if $\{Z_j\}_{j=1}^k$ are the irreducible components of Z of codimension two, then $c_1^2(N_{\mathcal{F}})$ is cohomologous to $\sum_{j=1}^k \mu_j [Z_j]$, for suitable complex

numbers μ_j (the Baum-Bott residus along Z_j). By the $\partial\bar{\partial}$ -lemma we therefore have

$$\omega \wedge \omega = \sum_{j=1}^k \mu_j \delta_{Z_j} + i\partial\bar{\partial}S$$

for a suitable current S of bidegree $(1,1)$.

If $n \geq 3$, the components Z_j are positive dimensional, and being disjoint from \mathcal{M} they are necessarily contained in the exceptional subset $Y \subset X \setminus \mathcal{M}$. Hence, under the canonical contraction $\pi : X \rightarrow X_0$, which collapses each connected component of Y to a point, each Z_j is also collapsed to a point, whence the direct image by π of the current δ_{Z_j} is vanishing. That is,

$$\pi_*(\omega \wedge \omega) = i\partial\bar{\partial}S_0$$

where $S_0 = \pi_*(S)$.

This seems a quite strange and unlikely situation. Indeed, ω is a Kähler form (here we are assuming $N_{\mathcal{F}}$ ample on the full X , not only around \mathcal{M}), and we find unlikely that by a modification the strictly positive $(2,2)$ -form $\omega \wedge \omega$ may become $\partial\bar{\partial}$ -exact. This is certainly not the case if X_0 is, as X , projective. However, generally speaking X_0 is only a Moishezon space [Pet], non projective, and on Moishezon spaces we may have nontrivial positive currents which are $\partial\bar{\partial}$ -exact; but usually (in the examples we know) these currents do not arise from powers of a Kähler form. Note that the current $\pi_*(\omega^{\wedge n})$ is a strictly positive measure, hence it cannot be $\partial\bar{\partial}$ -exact. Also, it is easy to see that $\pi_*(\omega)$ cannot be $\partial\bar{\partial}$ -exact, otherwise ω would be cohomologous to a divisor with support in Y , an evident absurdity. However, the non- $\partial\bar{\partial}$ -exactness of the intermediate powers $\pi_*(\omega^{\wedge k})$, $2 \leq k \leq n-1$, seems less evident.

This difficulty does not exist in [Lin] because there $X \setminus \mathcal{M}$ is not only strongly pseudoconvex but even Stein, and thus there is nothing to contract.

REFERENCES

- [BdB] E. Bedford, P. De Bartolomeis, *Levi flat hypersurfaces which are not holomorphically flat*, Proc. AMS 81 (1981), 575-578
- [BLM] C. Bonatti, R. Langevin, R. Moussu, *Feuilletages de $\mathbb{C}P^n$: de l'holonomie hyperbolique pour les minimaux exceptionnels*, IHES Publ. Math. 75 (1992), 123-134
- [Bri] J. Brinkschulte, *The $\bar{\partial}$ -problem with support conditions on some weakly pseudoconvex domains*, Ark. Mat. 42 (2004), 259-282
- [Bru] M. Brunella, *Mesures harmoniques conformes et feuilletages du plan projectif complexe*, preprint (2006)
- [CLS] C. Camacho, A. Lins Neto, P. Sad, *Minimal sets of foliations on complex projective spaces*, IHES Publ. Math. 68 (1988), 187-203
- [Der] B. Deroin, *Hypersurfaces Levi-plates immergées dans les surfaces complexes de courbure positive*, Ann. Sci. ENS 38 (2005), 57-75
- [Fuj] R. Fujita, *Domaines sans point critique intérieur sur l'espace projectif complexe*, J. Math. Soc. Japan 15 (1963), 443-473
- [Lin] A. Lins Neto, *A note on projective Levi flats and minimal sets of algebraic foliations*, Ann. Inst. Fourier 49 (1999), 1369-1385
- [Ohs] T. Ohsawa, *On real-analytic Levi-flats in complex tori*, preprint (2006)
- [Pet] Th. Peternell, *Pseudoconvexity, the Levi problem and vanishing theorems*, Several complex variables VII, Enc. Math. Sci. 74 (1994), 221-257
- [Pom] Ch. Pommerenke, *Boundary behaviour of conformal maps*, Springer Grundlehren 299 (1992)
- [ShB] N. I. Shepherd-Barron, *Miyaoka's theorems on the generic seminegativity of T_X* , Flips and abundance for algebraic threefolds, Astérisque 211 (1992), 103-115

- [Siu] Y. T. Siu, *Nonexistence of smooth Levi-flat hypersurfaces in complex projective spaces of dimension ≥ 3* , Ann. of Math. 151 (2000), 1217-1243
- [Suw] T. Suwa, *Indices of vector fields and residues of singular holomorphic foliations*, Hermann Actualités Mathématiques (1998)
- [Tak] A. Takeuchi, *Domaines pseudoconvexes infinis et la métrique riemannienne dans un espace projectif*, J. Math. Soc. Japan 16 (1964), 159-181

MARCO BRUNELLA, IMB - CNRS UMR 5584, 9 AVENUE SAVARY, 21078 DIJON, FRANCE